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On the Fine and Rough Theory of Lagrange Type Interpolation of Higher Order[1]

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This paper establishes the fine and rough theory of Lagrange type interpolation of higher order, which extends the results given by P. Erdős and P. Turán in *Acta Math. Acad. Sci. Hungar.* **6** (1955), 47–65. © 2000 Academic Press

1. INTRODUCTION AND MAIN RESULTS

Let $m, n, N \in \mathbb{N}$. Let us consider a triangular matrix X of nodes

$$1 \geqslant x_{n1} > x_{n2} > \cdots > x_{nn} \geqslant -1, \qquad n = 1, 2, ... \tag{1.1}$$

and sometimes for simplicity omit the superfluous notations. Denote by $\mathbf{P}_N$ the set of polynomials of degree at most N and by A_{jk} the fundamental polynomials for Hermite interpolation, i.e., $A_{jk} \in \mathbf{P}_{mn-1}$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\,\delta_{kq}, \qquad p, j = 0, 1, ..., m-1, \qquad q, k = 1, 2, ..., n.$$

The Hermite–Fejér interpolation for $f \in C[-1, 1]$ is given by

$$H_{nm}(f, x) := H_{nm}(X, f, x)$$
$$:= \sum_{k=1}^{n} f(x_k)\,A_k(x) \qquad (A_k := A_{0k},\ k = 1, 2, ..., n)$$

and the Hermite interpolation for $f \in C^{m-1}[-1, 1]$ is given by

$$H_{nm}^{*}(f, x) := H^{*}(X, f, x) := \sum_{j=0}^{m-1} \sum_{k=1}^{n} f^{(j)}(x_k)\,A_{jk}(x).$$

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To give an explicit formula for A_{jk} set

$$\omega_n(x)=(x-x_1)(x-x_2)\cdots(x-x_n),$$

$$\ell_k(x)=\frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}, \qquad k=1, 2, ..., n,$$

$$b_{\nu k}=\frac{1}{\nu!}\,[\ell_k(x)^{-m}]^{(\nu)}_{x=x_k}, \qquad \nu=0, 1, ..., m-1, \qquad k=1, 2, ..., n,$$

$$B_{jk}(x)=\sum_{\nu=0}^{m-j-1} b_{\nu k}(x-x_k)^\nu, \qquad j=0, 1, ..., m-1, \qquad k=1, 2, ..., n.$$

Then by [7, Lemma 1] we have

$$A_{jk}(x)=\frac{1}{j!}\,(x-x_k)^j\, B_{jk}(x)\, \ell_k(x)^m, \qquad j=0, 1, ..., m-1, \qquad k=1, 2, ..., n. \tag{1.2}$$

Put

$$\mu_n(x):=\mu_n(X, x):=\sum_{k=1}^{n} |A_k(X, x)|.$$

In what follows $\bigcirc(1)$, c, c_1, ... will denote positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas.

When m is odd H_{nm} is called Lagrange type interpolation, for which J. Szabados proved a Faber type theorem (the original statement is more general):

THEOREM A [7, Theorem 1]. *If m is odd then*

$$\mu_n(X)=\|H_{nm}(X)\|=\left\|\sum_{k=1}^{n} |A_k(X)|\right\| \geqslant c \ln n,$$

where $\|\cdot\|$ stands for the uniform norm on $[-1, 1]$.

That is to say, if m is odd then for any matrix X there must exist an $f\in C[-1, 1]$ such that

$$\limsup_{n\to\infty} \|H_{nm}(X, f)-f\|>0. \tag{1.3}$$

But as P. Erdős and P. Turán discovered in their paper [4], for Lagrange interpolation ($m=1$) we can claim the convergence-divergence behaviour of $H_{n1}(X, f)$ for certain classes of functions according to the order of

$\mu_n(X)$. This leads to the so called fine and rough theory. The aim of this paper is to extend this theory to Lagrange type interpolation of higher order ($m \geqslant 3$). We write this theory as the following three theorems, in which the results for $m=1$ are the same as those given by P. Erdős and P. Turán in [4]. We mention that there is no rough theory for Hermite–Fejér type interpolation (i.e., for even m), which is proved by P. Vértesi for $m=2$ [9] and by the author for $m \geqslant 4$ [6].

THEOREM 1. *Let m be an odd integer and $\mu_n(X) \sim n^\delta$, $0<\delta<1$. If $\delta<\gamma\leqslant 1$, then*

$$\lim_{n\to\infty} \|H_{nm}(X, f)-f\| = 0 \tag{1.4}$$

holds for every $f \in \mathrm{Lip}\,\gamma$.

THEOREM 2. *Let m be an odd integer and $\mu_n(X) \sim n^\delta$, $0<\delta<1$. If*

$$0<\gamma<\frac{m\delta}{\delta+2m}, \tag{1.5}$$

then there exists an $f \in \mathrm{Lip}\,\gamma$ such that (1.3) holds.

THEOREM 3. *Let m be an odd integer. If*

$$\frac{(2m-1)\,\delta}{\delta+4m-2}<\gamma<\delta<1, \tag{1.6}$$

then there exists a matrix $X=Y_1$ such that $\mu_n(Y_1)\sim n^\delta$ and

$$\lim_{n\to\infty} \|H_{nm}(Y_1, f)-f\| = 0 \tag{1.7}$$

holds for every $f \in \mathrm{Lip}\,\gamma$.

On the other hand, if $0<\delta<1$ then there exists a matrix $X=Y_2$ such that $\mu_n(Y_2)\sim n^\delta$ and an $f \in \mathrm{Lip}\,\delta$ such that

$$\limsup_{n\to\infty} \|H_{nm}(Y_2, f)-f\| > 0. \tag{1.8}$$

The paper is organized as follows. In the next section we state some auxiliary lemmas. In Section 3 we give the proofs of the theorems, using many ideas of [4] and [8, pp. 7–30]. In the last section some remarks are given.

2. AUXILIARY LEMMAS

For convenience we state some known results.

Let $d_1 = x_1 - x_2$, $d_n = x_{n-1} - x_n$, $d_k = \max\{|x_k - x_{k-1}|, |x_k - x_{k+1}|\}$, $2 \leqslant k \leqslant n-1$.

LEMMA A [5, Theorem 2.1]. *If $m-j$ is odd and $j < i \leqslant m-1$ then*

$$B_{jk}(x) \geqslant c_1 \left| \frac{x - x_k}{d_k} \right|^{i-j} |B_{ik}(x)|, \qquad x \in \mathbb{R}, \qquad 1 \leqslant k \leqslant n, \tag{2.1}$$

where c_1 is a positive constant depending only on m.

Moreover, if m is odd then

$$|A_{ik}(x)| \leqslant c_2 \, d_k^i \, |A_k(x)|, \qquad i \geqslant 0, \qquad x \in \mathbb{R}, \qquad 1 \leqslant k \leqslant n, \tag{2.2}$$

where c_2 is a positive constant depending only on m.

LEMMA B [5, Lemma 4.1]. *Let $Q_k \in \mathbf{P}_n$, $k = 1, 2, ..., N$, and $1 \geqslant y_1 > y_2 > \cdots > y_N \geqslant -1$. If*

$$\left\| \sum_{k=1}^{N} |(x - y_k) \, Q_k(x)| \right\| = v_n \tag{2.3}$$

and

$$\sum_{k=1}^{N} |Q_k(y_j)| \leqslant w_n, \qquad j = 1, 2, ..., N, \tag{2.4}$$

then

$$\left\| \sum_{k=1}^{N} |Q_k| \right\| \leqslant 2(n^2 v_n + w_n). \tag{2.5}$$

To state another important result given in [8, pp. 7–17] let us introduce some definitions and notations.

Let ω be a modulus of continuity and $C(\omega) := \{f \in C[-1, 1] : \omega(f, t) \leqslant c(f) \, \omega(t)\}$, where $c(f)$ is a positive constant depending on f. Let $\{E_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ be bounded linear operators from $C(\omega)$ into $C(\omega)$. Further we define the following conditions.

Condition I. There exist functions $g_n(x)$, $h_n(x)$, a sequence $\mathbb{M} := \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$, points $z_n \in [-1, 1]$, $n = 1, 2, ...,$ and constants c_3, $c_4 > 0$ such that

(I1) $g_n \in C(\omega)$, $n = 1, 2, ...,$

(I2) $E_n(g_n, z_n) \geqslant c_3 h_n(z_n) > 0$, $n \in \mathbb{M}$,

(I3) $E_n(g_N, z_n) \leqslant c_4 h_n(z_n)$, $n > N$, $n, N \in \mathbb{M}$.

Condition II.

(II1) There exists a sequence of numbers δ_n, $0 < \delta_n \leqslant 1$, such that $\lim_{n \to \infty} \delta_n = 0$ and

$$\omega(g_n, t) \leqslant c_5 t/\delta_n, \qquad n = 1, 2, ...,$$

(II2) $|g_n(t)| \leqslant c_6$,

(II3) $\lim_{t \to +0} \omega(t)/t = \infty$.

Condition III. We have $\|F_n\| \leqslant c_7$, $n \in \mathbb{N}$, and $\lim_{k \to \infty} h_{m_k}(z_{m_k}) = \infty$.

Then we have

Lemma C [8, Theorem 1.1, p. 10]. *If Conditions* I, II, *and* III *are true then there exists a function* $f \in C(\omega)$ *and a sequence of indices* $0 < n_1 < n_2 < \cdots$ *such that*

$$E_n(f, z_n) - F_n(f, z_n) > \omega(\delta_n)\, h_n(z_n), \qquad n = n_1, n_2, ... \tag{2.6}$$

Remark. The original statement of Theorem 1.1 in [8] is more general and complicated, but we only need its special case (Lemma C); here Conditions I, II, and III are Conditions A, B1, and C1 in [8, pp. 9–16], respectively.

Lemma 1. *If* $m - j$ *is odd and*

$$\|B_{jk} \ell_k^m\| \leqslant v_n, \qquad k = 1, 2, ..., n, \tag{2.7}$$

then

$$d_k \leqslant c_8\, \Delta_n(x_k) \ln^4(n v_n), \qquad k = 1, 2, ..., n, \tag{2.8}$$

and

$$\|\ell_k\| = \bigcirc(1)\, v_n^{1/m} \ln^4(n v_n), \qquad k = 1, 2, ..., n, \tag{2.9}$$

where

$$\Delta_n(x) = \frac{(1 - x^2)^{1/2}}{n} + \frac{1}{n^2}.$$

Proof. (2.7) by (2.1) gives

$$\left|\left(\frac{x-x_k}{d_k}\right)^{m-j-1} \ell_k(x)^m\right| = \bigcirc(1)\, v_n, \qquad k=1, 2, ..., n. \tag{2.10}$$

Thus

$$|(x-x_k)^m\, \ell_k(x)^m| = \bigcirc(1)\, v_n, \qquad k=1, 2, ..., n,$$

or

$$|(x-x_k)\, \ell_k(x)| = \bigcirc(1)\, v_n^{1/m}, \qquad k=1, 2, ..., n.$$

Applying Lemma B yields $\|\ell_k\| = \bigcirc(1)\, n^2 v_n^{1/m}$, $k=1, 2, ..., n$. By an estimation of P. Erdős [3] we obtain

$$|\theta_{k+1}-\theta_k| = \bigcirc(1)\, \frac{\ln n \ln(nv_n)}{n} = \bigcirc(1)\, \frac{\ln^2(nv_n)}{n}.$$

For $0 \leqslant k \leqslant n$ we have (with $x_k = \cos\theta_k$)

$$\begin{aligned}
&|\cos\theta_{k+1} - \cos\theta_k| \\
&\quad = \left|2\sin\left(\theta_k + \frac{\theta_{k+1}-\theta_k}{2}\right)\sin\frac{\theta_{k+1}-\theta_k}{2}\right| \\
&\quad \leqslant \left|(\theta_{k+1}-\theta_k)\sin\left(\theta_k + \frac{\theta_{k+1}-\theta_k}{2}\right)\right| \\
&\quad = \left|(\theta_{k+1}-\theta_k)\left(\sin\theta_k \cos\frac{\theta_{k+1}-\theta_k}{2} + \cos\theta_k \sin\frac{\theta_{k+1}-\theta_k}{2}\right)\right| \\
&\quad \leqslant (\theta_{k+1}-\theta_k)(\sin\theta_k + \theta_{k+1}-\theta_k).
\end{aligned}$$

Hence (2.8) follows.

Let us prove (2.9). Let k be fixed and $|\ell_k(\xi)| = \|\ell_k\|$, $\xi \in [-1, 1]$. Then (2.10) becomes

$$\left|\left(\frac{\xi-x_k}{d_k}\right)^{m-j-1} \ell_k(\xi)^m\right| = \bigcirc(1)\, v_n. \tag{2.11}$$

We distinguish two cases. Assume $n \geqslant 3$, in this case $u_n := \ln^4(nv_n) \geqslant \ln^4 3 \geqslant 1$, because $v_n \geqslant 1$.

Case 1. $|\xi - x_k| \geqslant (1/4c_8)\, d_k u_n^{-m/(m-1)}$, where c_8 is given in (2.8). In this case (2.11) yields (2.9).

Case 2. $|\xi - x_k| < (1/4c_8)\, d_k u_n^{-m/(m-1)}$. By the mean value theorem for the derivative

$$\|\ell_k\| - 1 \leqslant |\ell_k(\xi) - \ell_k(x_k)| = |\ell'_k(\eta)(\xi - x_k)|. \tag{2.12}$$

If $\sin\theta_k \leqslant 1/n$, then by (2.8) $d_k \leqslant 2c_8 u_n/n^2$ and hence $|\xi - x_k| \leqslant 1/(2n^2)$. Thus by (2.12)

$$\|\ell_k\| - 1 \leqslant n^2\, \|\ell_k\| \frac{1}{2n^2} = \frac{1}{2}\, \|\ell_k\|,$$

which implies $\|\ell_k\| \leqslant 2$, being more than stated in (2.9).

If $\sin\theta_k > 1/n$, then by (2.8)

$$d_k \leqslant \frac{2c_8 u_n \sin\theta_k}{n} \leqslant 2c_8 u_n \sin^2\theta_k$$

and hence

$$|\xi - x_k| \leqslant \frac{\sin^2\theta_k}{2u_n^{1/(m-1)}}. \tag{2.13}$$

On the other hand, a simple geometrical observation shows ($\xi = \cos\theta$)

$$\begin{aligned} |\xi - x_k| &= |\cos\theta - \cos\theta_k| = \left|\int_{\theta_k}^{\theta} \sin t\, dt\right| \geqslant \left|\int_0^{\theta-\theta_k} \sin t\, dt\right| \\ &= 1 - \cos(\theta - \theta_k) = 2\sin^2\frac{\theta-\theta_k}{2} \geqslant \frac{2}{\pi^2}(\theta - \theta_k)^2. \end{aligned}$$

Hence by (2.13) we get

$$|\theta - \theta_k| \leqslant \frac{\pi \sin\theta_k}{2u_n^{1/(2m-2)}}. \tag{2.14}$$

Let $\eta = \cos\tau$, where η is given in (2.12). Then $|\sin\tau - \sin\theta_k| \leqslant |\tau - \theta_k| \leqslant |\theta - \theta_k|$ and hence

$$\sin\tau \geqslant \sin\theta_k - |\theta - \theta_k|. \tag{2.15}$$

Next, since for $n \geqslant e^{\pi^{(m-1)/2}}$ we have $u_n > \pi^{2(m-1)}$, (2.14) and (2.15) imply $\sin\tau \geqslant \frac{1}{2}\sin\theta_k$. Thus for $n \geqslant e^{\pi^{(m-1)/2}}$ by (2.12) and (2.13)

$$\|\ell_k\| - 1 \leqslant |\ell'_k(\eta)(\xi - x_k)| \leqslant \frac{n\, \|\ell_k\|}{\sin\tau} \cdot \frac{\sin\theta_k}{2nu_n^{1/(m-1)}} \leqslant \frac{\|\ell_k\|}{u_n^{1/(m-1)}} \leqslant \frac{\|\ell_k\|}{2},$$

which again implies $\|\ell_k\| \leqslant 2$, being more than needed in (2.9). ∎

Let $\varphi(x) = (1-x^2)^{1/2}$. The weighted modulus of continuity of f defined by Ditzian and Totik [1] is given by

$$\omega_\varphi(f, t) := \sup_{0<h\leqslant t} \|f(\cdot + h\varphi(\cdot)/2) - f(\cdot - h\varphi(\cdot)/2)\|,$$

where the expression inside $\|\cdot\|$ is taken to be zero if $x \pm h\varphi(x)/2 \notin (-1, 1)$.

LEMMA D [2]. *For every $f \in C^r[-1, 1]$ $(r \geqslant 0)$ there exists a polynomial $P_n \in \mathbf{P}_n$ such that for all $x \in [-1, 1]$*

$$|f^{(j)}(x) - P_n^{(j)}(x)| \leqslant c\, \Delta_n(x)^{r-j}\, \omega_\varphi\left(f^{(r)}, \frac{1}{n}\right), \qquad 0 \leqslant j \leqslant r, \tag{2.16}$$

$$|P_n^{(j)}(x)| \leqslant c\, \Delta_n(x)^{r-j}\, \omega_\varphi\left(f^{(r)}, \frac{1}{n}\right), \qquad j > r. \tag{2.17}$$

P. Vértesi and Y. Xu in [11] give the rate of convergence in terms of the modulus $\omega_\varphi(f, t)$ of Ditzian and Totik for truncated Hermite interpolation on the zeros of the Jacobi polynomials. Developing and properly modifying their ideas we can give the rate of convergence in terms of the modulus $\omega_\varphi(f, t)$ of Ditzian and Totik for Lagrange type interpolation on an arbitrary system of nodes. That is the following

LEMMA 2. *If $f \in C[-1, 1]$ then*

$$\|H_{nm}(X, f) - f\| = \bigcirc(1)\, \mu_n(X)\{\ln[n\mu_n(X)]\}^{4(m-1)}\, \omega_\varphi\left(f, \frac{1}{n}\right). \tag{2.18}$$

Proof. Let $P_n \in \mathbf{P}_n$ be given in Lemma D. Then (2.16) and (2.17) with $r = 0$ hold. By (2.16) and [1, Theorem 7.1.1] $\|f - P_n\| \leqslant c\omega_\varphi(f^{(r)}, 1/n)$. Meanwhile, since $P_n(x) = H^*_{nm}(P_n, x)$, we have

$$\begin{aligned} \|H_{nm}(f) - f\| &\leqslant \|f - P_n\| + \left\| \sum_{k=1}^{n} [P_n(x_k) - f(x_k)]\, A_k \right\| \\ &\quad + \sum_{j=1}^{m-1} \left\| \sum_{k=1}^{n} P_n^{(j)}(x_k)\, A_{jk} \right\| \\ &= \bigcirc(1)[1 + \mu_n(X)]\, \omega_\varphi\left(f, \frac{1}{n}\right) + \sum_{j=1}^{m-1} \left\| \sum_{k=1}^{n} P_n^{(j)}(x_k)\, A_{jk} \right\|. \end{aligned} \tag{2.19}$$

By (2.2), (2.8), and (2.17)

$$\left\| \sum_{k=1}^{n} P_n^{(j)}(x_k)\, A_{jk} \right\|$$
$$= \bigcirc(1)\, \omega_\varphi\left(f, \frac{1}{n}\right) \left\| \sum_{k=1}^{n} \Delta_n(x_k)^{-j}\, d_k^j A_k \right\|$$
$$= \bigcirc(1)\, \omega_\varphi\left(f, \frac{1}{n}\right) \left\| \sum_{k=1}^{n} \{\Delta_n(x_k)^{-1}\, \Delta_n(x_k) \ln^4[n\mu_n(X)]\}^j A_k \right\|$$
$$= \bigcirc(1)\, \mu_n(X) \ln^{4j}[n\mu_n(X)]\, \omega_\varphi\left(f, \frac{1}{n}\right),$$

which together with (2.19) gives (2.18). ∎

3. PROOFS OF THEOREMS

3.1. *Proof of Theorem* 1. Since $f \in \operatorname{Lip} \gamma$ implies

$$\omega\left(f, \frac{1}{n}\right) = \bigcirc(n^{-\gamma}), \tag{3.1}$$

using the inequality $\omega_\varphi(f, 1/n) \leqslant \omega(f, 1/n)$ (2.18) directly yields $\|H_{nm}(f) - f\| = \circ(1)$. ∎

3.2. *Proof of Theorem* 2. By the same argument as that of Theorem 1.1 and Statement 1 in [8, pp. 9–19] let $\omega(t) = t^\gamma$, $E_n(f, x) = H_{nm}(f, x)$, $F_n = I$, $h_n(x) = \mu_n(X, x)$, $\mu_n(z_n) = \mu_n(X)$, and

$$g_n(x) = \begin{cases} \operatorname{sgn} A_k(z_n), & x = x_k, \quad 1 \leqslant k \leqslant n, \\ \text{linear}, & x \in [x_{k+1}, x_k], \quad 1 \leqslant k \leqslant n-1, \\ \text{constant}, & x \in [-1, x_n] \cup [x_1, 1]. \end{cases} \tag{3.2}$$

Then $|g_n(x)| \leqslant 1$, $g_n \in C(\omega)$, $H_{nm}(g_n, z_n) = \mu_n(X)$, and $|H_{nm}(g_N, z_n)| \leqslant \mu_n(X)$. Thus Condition I holds. To prove Condition II let $\delta_n = \rho_n := \min_{1 \leqslant k \leqslant n-1} |x_{k+1} - x_k|$. Then $\omega(g_n, t) \leqslant 2t/\delta_n$, which implies Condition II. Since $\mu_n(z_n) = \mu_n(X) \geqslant c \ln n$, Condition III is true. By Lemma C we conclude that

$$\|H_{nm}(f) - f\| \geqslant \mu_n(X)\, \omega(\rho_n) \tag{3.3}$$

holds for $n = n_1, n_2, \ldots$. Assume that $\rho_n = x_r - x_{r+1}$. Then by (2.9) and the mean value theorem for the derivative

$$\frac{1}{\rho_n} = \frac{1}{x_r - x_{r+1}} = \frac{\ell_r(x_r) - \ell_r(x_{r+1})}{x_r - x_{r+1}} = |\ell_r'(\xi)| \leqslant n^2 \, \|\ell_r\| \leqslant cn^{2+\delta/m} \ln^4 n.$$

Hence

$$\rho_n \geqslant (cn^{2+\delta/m} \ln^4 n)^{-1}.$$

Using the relation $\mu_n(X) \sim n^\delta$ we obtain

$$\mu_n(X)\,\omega(\rho_n) \geqslant \mu_n(X)\,\omega((cn^{2+\delta/m} \ln^4 n)^{-1}) \geqslant c^{-1} n^{\delta - \gamma(2+\delta/m)} (\ln n)^{-4}.$$

By virtue of (1.5) and (3.3) we get (1.3). ▮

3.3. *Proof of Theorem* 3. First, let us prove the first half of the theorem. Following the line of the proof of the first part of Statement 3 in [8, pp. 19–25] let $1 \leqslant t_n < 3$, $n = 1, 2, \ldots$, and let $T_n(x)$ denote the nth Chebyshev polynomial of the first kind, so in this case let T denote X. If x_k's stand for the zeros of $T_n(x)$ then we consider a new matrix Y:

$$y_k = \begin{cases} \cos \dfrac{t_n \pi}{2n}, & k = 1, \\ x_k, & 2 \leqslant k \leqslant n. \end{cases}$$

The corresponding fundamental polynomials of Lagrange interpolation $\ell_k(T, x)$ and $\ell_k(Y, x)$ satisfy the relations [8, (1.18) and (1.19), pp. 19]

$$\begin{cases} \ell_1(Y, x) = \dfrac{T_n(x)}{x - x_1} \cdot \dfrac{y_1 - x_1}{T_n(y_1)}, \\ \ell_k(Y, x) = \ell_k(T, x)\, \dfrac{x - y_1}{x - x_1} \cdot \dfrac{x_k - x_1}{x_k - y_1}, \qquad 2 \leqslant k \leqslant n. \end{cases} \tag{3.4}$$

We need the formulas [8, (1.20) and (1.34), pp. 20–22]

$$\left\| \sum_{k=3}^{n} |\ell_k(Y)| \right\| = \bigcirc(\ln n), \tag{3.5}$$

$$y_1 - x_2 \sim \frac{3 - t_n}{n^2}, \tag{3.6}$$

and [8, pp. 20–21]

$$\|\ell_k(Y)\| = \bigcirc(1), \qquad 3 \leqslant k \leqslant n. \tag{3.7}$$

By the same reason as that in [8, p. 20] since $Y = T$ when $t_n = 1$ (in this case $\mu_n(Y) = \mu_n(T) \sim \ln n$ [7, Theorem 2]) and $v_n(Y) \geqslant \|A_2(Y)\| \to \infty$ when $t_n \to 3$, for each n there exists a t_n such that, denoting by Y_1 such a matrix,

$$\mu_n(Y_1) \sim n^\delta, \qquad n = 1, 2, \ldots . \tag{3.8}$$

To prove our conclusion we must show that

$$\left\| \sum_{k=3}^{n} |A_k(Y_1)| \right\| = \bigcirc(\ln n), \tag{3.9}$$

$$\|A_1(Y_1) + A_2(Y_1)\| = \bigcirc(\ln n) \tag{3.10}$$

(it is enough to get an estimation $\bigcirc(\ln n)$ instead of $\bigcirc(1)$ (see Remark 4.2)), and if

$$\| |A_1(Y_1)| + |A_2(Y_1)| \| \geqslant \tfrac{1}{2}\mu_n(Y_1), \tag{3.11}$$

then

$$y_1 - x_2 \leqslant \frac{\bigcirc(1)}{n^2 \mu_n(Y_1)^{1/(2m-1)}}. \tag{3.12}$$

Let us calculate $b_{ik}(Y_1)$, $k \geqslant 2$. To this end we calculate $a_{ik}(Y_1)$:

$$\begin{aligned} a_{ik}(Y_1) := \sum_{v \neq k} \frac{1}{(y_v - x_k)^i} &= \frac{1}{(y_1 - x_k)^i} - \frac{1}{(x_1 - x_k)^i} + \sum_{v \neq k} \frac{1}{(x_v - x_k)^i} \\ &= \frac{1}{(y_1 - x_k)^i} - \frac{1}{(x_1 - x_k)^i} + a_{ik}(T) \\ &= \frac{1}{(y_1 - x_k)^i} - \frac{1}{(x_1 - x_k)^i} + \bigcirc(1) \left[\frac{n}{(1 - x_k^2)^{1/2}} \right]^i. \end{aligned}$$

Thus

$$|a_{ik}(Y_1)| = \begin{cases} \bigcirc(1) \left[\dfrac{n}{(1 - x_k^2)^{1/2}} \right]^i, & k \geqslant 3, \\ \bigcirc(1) \left[\dfrac{n^2}{3 - t_n} \right]^i, & k = 2. \end{cases} \tag{3.13}$$

Here the last formula follows from (3.6). Then by induction from the formula [7, (10)]

$$b_{ik} = \frac{m}{i} \sum_{v=1}^{i} a_{vk} b_{i-v,k}, \qquad i \geqslant 1$$

we have

$$|b_{ik}(Y_1)| = \begin{cases} \bigcirc(1) \left[\dfrac{n}{(1-x_k^2)^{1/2}} \right]^i, & k \geqslant 3, \\ \bigcirc(1) \left[\dfrac{n^2}{3-t_n} \right]^i, & k = 2. \end{cases} \tag{3.14}$$

By (1.2), (3.14), (3.4), and (3.7) for $k \geqslant 3$

$$\begin{aligned} |A_k(Y_1, x)| &= \bigcirc(1) \sum_{i=0}^{m-1} \left| \frac{n}{(1-x_k^2)^{1/2}} \cdot (x - x_k) \right. \\ &\quad \times \left. \frac{T_n(x)(x-y_1)(x_k - x_1)}{T'_n(x_k)(x-x_k)(x-x_1)(x_k-y_1)} \right|^i |\ell_k(Y_1, x)|^{m-i} \\ &= \bigcirc(1) \sum_{i=0}^{m-1} \left| \left[\frac{T_n(x)(x-y_1)}{(x-x_1)} \right] \left[\frac{x_k - x_1}{x_k - y_1} \right] \right|^i |\ell_k(Y_1, x)|^{m-i} \\ &= \bigcirc(1) \, |\ell_k(Y_1, x)|, \end{aligned}$$

which by (3.5) gives (3.9).

It is simple to prove (3.10). In fact, since $\sum_{k=1}^{n} A_k(Y_1, x) \equiv 1$, by (3.9)

$$\|A_1(Y_1) + A_2(Y_1)\| = \left\| 1 - \sum_{k=3}^{n} A_k(Y_1, x) \right\| \leqslant 1 + \left\| \sum_{k=3}^{n} |A_k(Y_1)| \right\| = \bigcirc(\ln n).$$

Finally, to prove (3.12) we observe first that by (3.8)–(3.10)

$$\|A_2(Y_1)\| \geqslant \tfrac{1}{5} \mu_n(Y_1) \tag{3.15}$$

for n large enough.

On the other hand, by (3.4), (3.6), and the well known property of the Chebyshev polynomials

$$\begin{aligned} |\ell_2(Y_1, x)| &= \left| \frac{T_n(x)(x-y_1)(x_2-x_1)}{T'_n(x_2)(x-x_2)(x-x_1)(x_2-y_1)} \right| \\ &= \frac{\bigcirc(1)}{3-t_n} \left| \frac{T_n(x)(x-y_1)}{n^2(x-x_2)(x-x_1)} \right| = \frac{\bigcirc(1)}{3-t_n}. \end{aligned}$$

Hence by (1.2), (3.6), and (3.14) we get

$$|A_2(Y_1, x)| = \bigcirc(1) \sum_{i=0}^{m-1} \left| \frac{n^2(x-x_2)}{3-t_n} \times \frac{T_n(x)(x-y_1)(x_2-x_1)}{T'_n(x_2)(x-x_2)(x-x_1)(x_2-y_1)} \right|^i |\ell_2(Y_1, x)|^{m-i}$$

$$= \frac{\bigcirc(1)}{(3-t_n)^{2m-1}},$$

which, coupled with (3.6) and (3.15), gives (3.12):

$$y_1 - x_2 = \bigcirc\left(\frac{3-t_n}{n^2}\right) = \frac{\bigcirc(1)}{n^2 \mu_n(Y_1)^{1/(2m-1)}}.$$

Now suppose that $f \in \operatorname{Lip} \gamma$ is arbitrary. Let $N = 1 + [n^{(1-\delta)/3}]$ and let $P \in \mathbf{P}_N$ be the best approximation to f from $\mathbf{P}_N$. Then noting that $P(x) = H^*_{nm}(Y_1, P, x)$.

$$|f(x) - H_{nm}(Y_1, f, x)| \leqslant |f(x) - P(x)| + \left| \sum_{k=1}^{n} [P(y_k) - f(y_k)] A_k(Y_1, x) \right|$$

$$+ \sum_{j=1}^{m} \sum_{k=1}^{n} |P^{(j)}(y_k) A_{jk}(Y_1, x)|$$

$$:= S_1 + S_2 + S_3. \tag{3.16}$$

Clearly, $S_1 = \bigcirc(n^{-\gamma(1-\delta)/3})$.

By the same argument as that in [8, pp. 24–25] we rewrite S_2 as

$$S_2 = \left| \left\{ \sum_{k=3}^{n} [P(y_k) - f(y_k)] A_k(Y_1, x) \right\} + \{[P(y_1) - f(y_1)][A_1(Y_1, x) + A_2(Y_1, x)]\} + \{[P(y_2) - P(y_1) + f(y_1) - f(y_2)] A_2(Y_1, x)\} \right|.$$

By (3.8)–(3.10), (3.12), and by the mean value theorem for the derivative

$$\begin{aligned}S_2 &= \bigcirc(n^{-\gamma(1-\delta)/3})\ln n + [\,\|P'\|\,|y_2-y_1| + \bigcirc(1)\,|y_2-y_1|^{\gamma}]\,\mu_n(Y_1)\\ &= \bigcirc(n^{-\gamma(1-\delta)/3})\ln n\\ &\quad + \bigcirc(1)\left[\frac{N^2\,\|P\|}{n^2\mu_n(Y_1)^{1/(2m-1)}} + \frac{1}{(n^2\mu_n(Y_1)^{1/(2m-1)})^{\gamma}}\right]\mu_n(Y_1)\\ &= \bigcirc(n^{-\gamma(1-\delta)/3})\ln n + \bigcirc(n^{-1}) + \bigcirc(n^{[(2m-1)\,\delta-\gamma(\delta+4mm-2)]/(2m-1)})\\ &= \circ(1),\end{aligned}$$

since $(2m-1)\,\delta/(\delta+4m-2)<\gamma$.

As for S_3 we use (2.2), (3.1), and Markov Inequality to get

$$\begin{aligned}\sum_{k=1}^{n}|P^{(j)}(y_k)\,A_{jk}(Y_1,x)| \leqslant \|P^{(j)}\|\sum_{k=1}^{n}|A_{jk}(Y_1,x)| &= \bigcirc(1)\,\|P^{(j)}\|\,D_n^j\mu_n(Y_1)\\ &= \bigcirc(1)\,N^{2j}n^{\delta-j}(\ln n)^{2j}\\ &= \bigcirc(1)\,n^{2j(1-\delta)/3+\delta-j}(\ln n)^{2j} = \circ(1), \qquad j\geqslant 1.\end{aligned}$$

So $S_3=\circ(1)$. This proves $\|f-H_{nm}(Y_1,f)\|=\circ(1)$.

Next, for the proof of the second half of the theorem let $N=[n/2]$ and

$$R_n^{(\alpha,\alpha)}(x) := \begin{cases} P_N^{(\alpha,\alpha)}(1-2x^2), & \text{if } n \text{ is even},\\ xP_N^{(\alpha+1,\alpha)}(1-2x^2), & \text{if } n \text{ is odd},\end{cases} \tag{3.17}$$

where $P_N^{(\alpha,\beta)}(x)$ are the Nth Jacobi polynomial with the normalization $P_N^{(\alpha,\beta)}(1)=\binom{N+\alpha}{N}$. $X=Z$ stands for the matrix consisting of the zeros (1.1) of $R_n^{(\alpha,\alpha)}(x)$. Denote by $\ell_k(Z,x)$ and $A_k(Z,x)$, $k=1,2,\ldots,n$, the fundamental polynomials for the Lagrange interpolation and the Hermite–Fejér interpolation with respect to Z, respectively. Then we have [6, Theorem 1 and Lemma 5] that

$$\mu_n(Z)\sim n^{m(\alpha+1/2)}, \qquad \alpha>-\tfrac{1}{2}, \tag{3.18}$$

and

$$|A_k(Z,1)|\geqslant cn^{m(\alpha+1/2)-1}, \qquad \alpha>-\frac{1}{2}, \qquad q:=\left[\frac{N}{2}\right]\leqslant k\leqslant p:=\left[\frac{3N}{4}\right]. \tag{3.19}$$

This shows that if we take $\alpha=\delta/m-\frac{1}{2}$ and denote by $Z=Y_2$ the corresponding matrix then $\mu_n(Y_2)\sim n^{\delta}$.

Finally, to apply Lemma C choose $g_n \in C[-1, 1]$ so that

$$g_n(x) = \begin{cases} 0, & x \geqslant x_{q-1} \quad \text{or} \quad x \leqslant x_{p+1}, \\ \operatorname{sgn} \ell_k(Y_2, 1), & x = x_k, \quad q \leqslant k \leqslant p, \\ \text{linear}, & x \in [x_{k+1}, x_k], \quad q-1 \leqslant k \leqslant p. \end{cases}$$

Then $|g_n(x)| \leqslant 1$ and

$$|g_n(x+t) - g_n(x)| \leqslant \frac{2t}{\min_{q-1 \leqslant k \leqslant p} |x_{k+1} - x_k|} = \bigcirc(nt),$$

that is to say, $g_n \in \text{Lip}\, 1$. Now let $E_n = H_{nm}$, $F_n = I$, $z_n = 1$, $\delta_n = 1/n$, $h_n(x) = n^\delta$, $\omega(t) = t^\delta$. We can again check that Conditions I, II, and III are valid. By Lemma C there is a function $f \in \text{Lip}\, \delta$ such that

$$\limsup_{n \to \infty} [H_{nm}(Y_2, f, 1) - f(1)] > 0. \quad \blacksquare$$

4. REMARKS

4.1. For $m > 1$ in Theorem 2 and the first half of Theorem 3 there is a gap

$$\frac{m\delta}{\delta + 2m} < \gamma \leqslant \frac{(2m-1)\,\delta}{\delta + 4m - 2}.$$

We believe that the first half of Theorem 3 remains true for this value of γ.

4.2. The proof of the preliminary result $\|\ell_1(Y) + \ell_2(Y)\| = \bigcirc(1)$ in [8, pp. 21–23] is complicated. But for the proof of the main result in stead of this fact it is enough to prove that $\|\ell_1(Y) + \ell_2(Y)\| = \bigcirc(\ln n)$, as stated in the present proof of Theorem 3, which may be easily observed by (3.5) and the identity $\sum_{k=1}^n \ell_k(Y, x) \equiv 1$. This argument simplifies the original proof.

4.3. For the proof of the second half of Theorem 3 we use a different method from that used in [8, pp. 25–30] for the case when $m = 1$. The reason is that for the matrix X used in that proof of [8] we obtain $\mu_n(X) \geqslant cn^{m-1} (> n^\delta)$, which obviously is not suitable for the present case when $m \geqslant 3$.

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